Algebra Qualifying Exam
Spring 2014

1. Let F_2 be the eld with 2 elements.

(a) Determine the Galois group of $x^5 + x^3 + x^2 + x + 1 \; 2 \; F_2[x]$.

(b) Exhibit a matrix of order 31 in $GL_5(F_2)$.

2. Suppose $p > 2$ is prime. Classify the groups of order $2p^2$.

3. Let F be a eld and suppose the minimal polynomial of $A \; 2 \; M_n(F)$ has degree n. Show that every matrix commuting with A has the form $p(A)$ for a polynomial $p(x) \; 2 \; F[x]$.

4. Suppose R is a Noetherian local ring and M is a nitely generated at R-module. Prove that M is free.

5. Fix a nite extension $K{=}F$ of sub elds of C, and $\;\; 2$ C.

(a) If is transcendental over F, prove that $[K(\;) : F(\;)] = [K : F]$.

(b) Find and example of F, K, and <u>algebraic</u> such that $[K(\;) : F(\;)]$ is not a divisor of $[K : F]$.

6. De ne $R = C[x; y]{=}(y^4 + x^2 \quad 1):$

(a) Show that R is an integral domain.

(b) Let K be the fraction eld of R. Show that K is Galois over $C(x)$, and compute the Galois group.

(c) For each prime ideal of $C[x]$, determine the number of primes of R lying above it, and nd generators for those primes.

7. Let $R = k[x]$ and $M = k[x; y]{=}(xy)$. Show that each of the following R-modules is isomorphic to a direct sum of cyclic factors, and describe the factors.

(a) $\mathrm{Tor}_1^R(M; R{=}(x))$.

(b) $\mathrm{Ext}_R^1(R{=}(x); M)$.

8. Let k be a eld. Find the Krull dimensions of

$$R = k[x; y; z]{=}(xz; yz);$$

$R{=}(x + y)$, and $R{=}(x + y + z)$.