

$$X = (x_1, s_1; \dots; x_n)$$

(3)

$$X \succeq Y \iff V^*(c_X, p_1, \dots, p_n) \geq V^*(c_Y, p_1, \dots, p_n)$$

Lemma 1. Let \succeq be a distribution-regret preference relation. Then \succeq admits a two-dimensional regret function $V^* : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R}$ and a regret functional V^* such that

$$\begin{aligned} X \succeq Y &\iff V^*(c_X, p_1, \dots, p_n; c_Y, p_1, \dots, p_n) \geq 0 \\ &\iff V^*(c_Y, q_1, \dots, q_m; c_X, q_1, \dots, q_m) \leq 0 \end{aligned}$$

where c_X and c_Y are the certainty equivalents of X and Y respectively.

Proof. Let $V^* : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R}$ be a two-dimensional regret functional, \succeq is a distribution-regret preference relation, $y \in \mathcal{D}$,

$$\begin{aligned} V^*(x, y) &= V^*(x, \delta_y), \\ V^*(x, y) &= V^*(x, \delta_y), \\ V^*(c_X, p_1, \dots, p_n; c_Y, p_1, \dots, p_n) &= V^*(c_X, \delta_Y, p_1, \dots, p_n) \end{aligned}$$

$$\begin{aligned} X \succeq Y &\iff X \succeq \delta_{c_Y} \\ &\iff V^*(c_X, \delta_{c_Y}, p_1, \dots, p_n) \geq 0 \\ &\iff V^*(c_X, p_1, \dots, p_n; c_Y, p_1, \dots, p_n) \geq 0 \quad \square \end{aligned}$$

Definition 4. \succeq distribution-regret based

$$: \mathcal{D} \times \mathcal{D} \rightarrow \mathfrak{R},$$

$$V : \rightarrow \mathfrak{R},$$

$$X \succeq Y \text{ iff } V(\Psi(X, c_Y)) \geq 0 \text{ iff } 0 \geq V(\Psi(Y, c_X)),$$

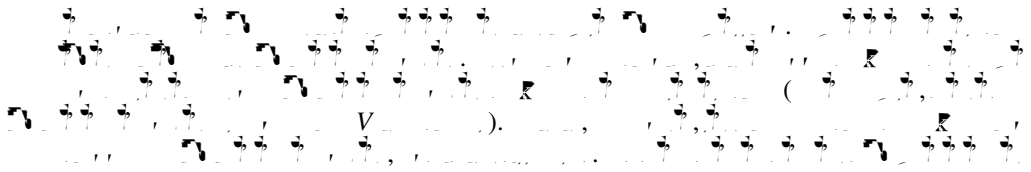
$$\Psi(X, c_Y) = ((x_1, c_Y), p_1; \dots; (x_n, c_Y), p_n)$$

$$X \delta_{c_Y} (Y), \Psi(Y, c_X)$$

$$Y \delta_{c_X} (X),$$

x

$$X \sim \delta_{cY} \implies V(n d \quad V$$



Proposition 3. *If the preference relation \succeq is consistent then it satisfies distribution regret.*

Proof. Let $Z \in \mathcal{L}$ with $Z = \delta_x$ for some $x \in \mathcal{D}$. Then $f(c_Z, \lambda) = 0$.¹⁰ Let $Y \in \mathcal{L}$ with $Y = \delta_x$ for some $x \in \mathcal{D}$. Then $f(c_Y, \lambda(Y)) = 0$.¹¹ Since \succeq is consistent, we have $\lambda(Z) \succeq \lambda(Y)$. This implies $U(\delta_x) \geq U(f(x, \lambda(Y)))$. Since $f(x, \lambda(Y)) = 0$, we have $U(\delta_x) \geq U(0)$. Since $x \in \mathcal{D}$, we have $f(x, \lambda(Y)) \in \mathcal{D}$.

$$\begin{aligned} f(X, \lambda(Y)) &= (f(x_1, \lambda(Y)), p_1; \dots; f(x_n, \lambda(Y)), p_n) \\ &= ((x_1, c_Y), p_1; \dots; (x_n, c_Y), p_n) \\ &= \Psi(X, c_Y) \end{aligned} \tag{4}$$

$$\begin{aligned} X \succeq Y \sim \delta_{c_Y} &\iff f(X, \lambda(Y)) \succeq f(Y, \lambda(Y)) \sim \delta_{f(c_Y, \lambda(Y))} = \delta_0 \\ &\iff U(f(X, \lambda(Y))) \geq U(f(Y, \lambda(Y))) = U(\delta_0) = 0 \end{aligned}$$

$$V(\Psi(X, c_Y)) = U(f(X, \lambda(Y))) = U(\Psi(X, c_Y))$$

$$\begin{aligned} X \succeq Y &\iff U(\Psi(X, c_Y)) \geq 0 \\ &\iff V(\Psi(X, c_Y)) \geq 0 \end{aligned}$$

□

Let $Z = (z_1, r_1; \dots; z_n, r_n)$ with $z_1 \leq \dots \leq z_n$. Then $f(c_Z, \lambda(Z)) = 0$.¹² Let $Y = (y_1, r_1; \dots; y_n, r_n)$ with $y_1 \leq \dots \leq y_n$. Then $f(c_Y, \lambda(Y)) = 0$.¹³ Since \succeq is consistent, we have $\lambda(Z) \succeq \lambda(Y)$. This implies $U(\delta_x) \geq U(f(x, \lambda(Y)))$. Since $f(x, \lambda(Y)) = 0$, we have $U(\delta_x) \geq U(0)$. Since $x \in \mathcal{D}$, we have $f(x, \lambda(Y)) \in \mathcal{D}$.

$$c_Z = u^{-1} \left(u(z_1)g(r_1) + \sum_{i=2}^n u(z_i) \left[g \left(\sum_{j=1}^i p_j \right) - g \left(\sum_{j=1}^{i-1} p_j \right) \right] \right)$$

$$f(x, \lambda) = u^{-1}(u(x) + \lambda)$$

$$f(c_Z, \lambda(Z)) = 0 \implies \lambda(Z) = -u(z_1)g(r_1) - \sum_{i=2}^n u(z_i) \left[g \left(\sum_{j=1}^i p_j \right) - g \left(\sum_{j=1}^{i-1} p_j \right) \right]$$

Since $\lambda(Z) \succeq \lambda(Y)$, we have $U(\delta_x) \geq U(f(x, \lambda(Y)))$. Since $f(x, \lambda(Y)) = 0$, we have $U(\delta_x) \geq U(0)$. Since $x \in \mathcal{D}$, we have $f(x, \lambda(Y)) \in \mathcal{D}$.

¹⁰ $0 \in [\mathcal{D}]$. For any $d \in [\mathcal{D}]$, $f(c_Z, \lambda(Z)) = d$ if and only if $d = 0$.

¹¹ $(x, x) = (x, c_{\delta_x}) = f(x, \lambda(\delta_x)) = 0$.

$$(x, c_Y) = f(x, \lambda(Y)) = u^{-1}(u(x) + \lambda(Y))$$



Example 3.

\succsim is a preference relation on \mathcal{I} , λ is a linear functional on \mathcal{I} , $\lambda(X) = 1$, $\lambda(Y) = 0$, $\lambda(Z) = 0$, $\lambda(X') = 1$, $\lambda(Y') = 0$, $\lambda(Z') = 0$.

$X, Y, Z \in \mathcal{I}$. $V(\Psi(X, c_Z)) = V(\Psi(Y, c_Z)) = 0$, $\alpha \in (0, 1)$, $V(\Psi(\alpha X + (1 - \alpha)Y, c_Z)) = 0$.

$$\Psi(X', c_{Z'}) =$$

$$[X] + \alpha \mu_X^+ = (1 + \alpha)\alpha \implies \alpha = \frac{-(1 - \mu_X^+) + \sqrt{(1 - \mu_X^+)^2 + 4 [X]}}{2} \quad (9)$$

$$[X] > 0 \implies (X < 0) > 0 \implies X > \delta_{[X]} \quad (10)$$

Since $\mu_X^+ > E[X] > 0$, $X \sim \delta_\alpha$, $\alpha > E[X]$.
 $f(-1, \lambda_0) = s$, $f(t, \lambda_0) = 0$, $z \geq t$, $-1 < s, t < 0$, λ_0

$$\left[\left(z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z} \right) \right] = t, \quad \left(z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z} \right) \sim (t, 1)$$

$$\left(f(z, \lambda_0), \frac{1+t}{1+z}; s, \frac{z-t}{1+z} \right) \sim (0, 1) \quad (10)$$

Proposition 4. *If the preference relation \succeq satisfies distribution regret with a commutative regret function Ψ , then it is consistent.*

Proof. $\forall d \in \mathbb{R}$, $(x, x) = d$, $x \in \mathcal{D}$. (4) $f(x, \lambda) = y$, $(x, y) = d - \lambda$. $(x, f(x, \lambda)) = d - \lambda$ (13)

$X \succeq Y$, $V(\Psi(X, c_Y)) \geq 0$. (13)

$$(c_X, f(c_X, \lambda)) = (c_Y, f(c_Y, \lambda)) = d - \lambda$$

$$(12) \quad (c_X, c_Y) = (f(c_X, \lambda), f(c_Y, \lambda)),$$

$$\Psi(\delta_{c_X}, c_Y) = \Psi(\delta_{f(c_X, \lambda)}, f(c_Y, \lambda))$$

$$\delta_{c_X} \geq \delta_{c_Y} \implies \delta_{f(c_X, \lambda)} \geq \delta_{f(c_Y, \lambda)}. \quad (13) \quad (12)$$

$$(x_i, c_X) = (f(x_i, \lambda), f(c_X, \lambda))$$

$$\Psi(X, c$$

5. Discussion

