

Algebra Qualifying Examination, Fall 2019

Instructions: This is a 3 hour examination. In the problems below, all rings are commutative with identity unless specified otherwise. This is a closed book exam, also no notes, searching the web, or otherwise consulting external sources. Good luck!

1. Let  $P$  be a finite  $p$ -group. Show that  $P$  is not cyclic if and only if  $P$  has a quotient isomorphic to  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ .

2. Let  $R$  be a commutative ring with unity.

(a) Let  $S$  be a non-empty saturated multiplicative set in  $R$ , i.e. if  $a, b \in R$ , then  $ab \in S$  if and only if  $a, b \in S$ . Show that  $R \setminus S$  (the complement of  $S$  in  $R$ ) is a union of prime ideals.

(b) Suppose that  $R$  is a principal ideal domain such that every non-zero prime ideal is maximal. Show that every non-zero prime ideal is principal and that  $R$  is a local PID.

(b) If  $F$  is an algebraically closed field and  $A \in GL_N(F)$  is of finite order, is  $A$  a scalar matrix?